Unit 4. Sensitivity Analysis

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Preliminary

This course is strongly based on the monography on Operations Research by Carter, Price and Rabadi [1], and in material obtained from different sources (quoted when needed through the slides).

Sensitivity Analysis: Summary

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Learning outcomes

- Understanding the concept of postoptimality analysis
- Understanding and solving LP problems requiring sensitvity analysis
- Understanding the matrix representation of the Simplex solution

The concept

- After an optimal solution is found, the analyst needs to review the problem parameters and the solution.
- This process is called **postoptimality analysis**:
  - confirming or updating problem parameters (cost and availabilities of activities and resources)
  - if changes need to be introduced in the original parameters, assessing their impact on the optimality of the solution.
- If changes are small, re-optimization may not be needed. LP solutions can be *very* sensitive — practical importance: decisions may change drastically.
- **Sensitivity analysis** is the study of the effect that types, ranges, and magnitude of changes in problem parameters have in the value of the objective function, *without the need to solve again the new linear problem*.

See, also:

https://home.ubalt.edu/ntsbarsh/opre640a/PARTVII.HTM

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Two types of parameter modifications

In a LP problem,

$$\begin{aligned} \max \quad & \mathbf{c}^t \mathbf{x} \\ \text{subject to} \quad & A\mathbf{x} \leq \mathbf{b}, \forall x_i \geq 0 \end{aligned}$$

one can have two situations: one may have interest in knowing what happens with modifications in the parameters c or modifications in the parameters b.
In addition, one can explore what occurs when adding an extra constraint or adding a new variable.

Remember some concepts

Remember:

- A binding constraint is active at the optimal solution: the left-hand side equals the right-hand side.
- A non-binding constraint has slack or surplus and does not influence the current optimal solution; therefore its shadow price is zero.
- A shadow price measures the rate of improvement of the objective when the RHS increases by one unit.
- In a maximization problem: positive shadow price $\Rightarrow$ increasing availability (relaxing the constraint) improves the objective until shadow price $= 0 \Rightarrow$ constraint not binding.
- In a minimization problem, a negative shadow price indicates that relaxing the constraint would worsen the objective.

Case 1: sensitivity with respect to c

Exercise 1

A farmer wants to minimize the cost of the food given to her lifestock. Two different types of nutrients A and B are needed by the animals, and she needs a minimum nutrition to be achieved.

	A	B	price
Feed 1	10	3	16
Feed 2	4	5	14
requirements	124	60	

How does the price change affects the minimum (consider c_1 and c_2 as new prices and compare the slope of the new objective function with the original one)? What happens with respect to basic vs non-basic variables in a general case?

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Case 2: sensitivity with respect to b

In such cases we will explore the coeficients of the objective function in the dual problem, instead.

Exercise 2

Given the LP problem

$$\begin{array}{rrcl} \text{minimize} & z = 16x_1 + 14x_2 & & \\ & 10x_1 + 4x_2 & \geq & 124 \\ \text{subject to} & 3x_1 + 5x_2 & \geq & 60 \\ & x_1, x_2 & \geq & 0 \end{array}$$

Explore the sensitivity of the minimum value of z with respect to the parameters in the RHS of the contraints. Hint: consider the dual problem.

Allowable Ranges and Sensitivity

- The allowable range of an objective coefficient is the interval within which that change does not alter the optimal basis: therefore, if a coefficient changes within this interval, the optimal solution does not change.
- The allowable range for the right-hand side (RHS) is the range over which the shadow prices remain valid; outside this interval the basis must be recalculated.

Shadow price is the solution of the dual problem

Solving the dual problem gives a lot of insight into the actual situation.

Exercise 3

* In a company that manufactures two types of bikes, A and B, the owners want to maximize the benefits, taking into account that the production depends on a restricted amount of titanium, the time the machines can devote to the work and the manpower, which are all given below:

	A	B	limit
Titanium	50	30	2000
Machine time	6	5	300
Labor	3	5	200
price	50	60	

Shadow price is the solution of the dual problem

Primal problem (production of bikes A and B):

$$\max_{x_A, x_B \geq 0} \; 50x_A + 60x_B$$

$$\text{s.t.} \begin{cases} 50x_A + 30x_B \leq 2000 & \text{(Titanium)} \\ 6x_A + 5x_B \leq 300 & \text{(Machine time)} \\ 3x_A + 5x_B \leq 200 & \text{(Labor)} \end{cases}$$

Vertices and optimum:

$$(x_A, x_B) = (25, 25), \quad Z_{\max} = 2750$$

Binding constraints: (1) Titanium, (3) Labor.

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Dual formulation

Dual problem:

$$\min_{y_1, y_2, y_3 \geq 0} 2000y_1 + 300y_2 + 200y_3$$

$$\text{s.t.} \begin{cases} 50y_1 + 6y_2 + 3y_3 \geq 50 & \text{(A)} \\ 30y_1 + 5y_2 + 5y_3 \geq 60 & \text{(B)} \end{cases}$$

Interpretation: y_i = marginal value (shadow price) of resource i:

- y_1: Titanium
- y_2: Machine time
- y_3: Labor

Solving the dual problem

From the primal optimum:

- $x_A > 0$, $x_B > 0 \Rightarrow$ dual constraints (A) and (B) are binding, so... **equalities**.
- Constraints (1) and (3) binding $\Rightarrow y_1 \geq 0$, $y_3 \geq 0$.
- Constraint (2) slack $\Rightarrow y_2 = 0$.

$$\begin{cases} 50y_1 + 3y_3 = 50 \\ 30y_1 + 5y_3 = 60 \end{cases} \Rightarrow y_1 = \frac{7}{16} = 0.4375, \quad y_3 = \frac{75}{8} = 9.375, \quad y_2 = 0.$$

$$Z_D = 2000y_1 + 300y_2 + 200y_3 = 2750 = Z_P.$$

Strong duality holds.

Interpretation of shadow prices

Resource	Dual variable	Interpretation
Titanium	$y_1 = 0.4375$	Profit increases €0.4375 per extra unit
Machine time	$y_2 = 0$	Not limiting $\Rightarrow$ no effect
Labor	$y_3 = 9.375$	Profit increases €9.375 per extra unit

Economic meaning:

- Only scarce resources (binding constraints) have positive shadow prices.
- Shadow prices measure marginal worth of each resource.
- Dual solution explains how resources limit total profit.

Complementary slackness summary

- $x_j > 0 \Rightarrow$ corresponding dual constraint holds with equality.
- $y_i > 0 \Rightarrow$ corresponding primal constraint is binding.

	Primal constraint	Dual var.	Binding?
Titanium	$50x_A + 30x_B \leq 2000$	$y_1 = 0.4375$	Yes
Machine time	$6x_A + 5x_B \leq 300$	$y_2 = 0$	No
Labor	$3x_A + 5x_B \leq 200$	$y_3 = 9.375$	Yes

The dual gives the shadow prices directly — the economic value of each constraint.

Degeneracy and Alternative Optima

- Degeneracy: one or more basic variables have value zero in the optimal solution. This can lead to multiple optimal solutions or complicate sensitivity analysis.
- Sensitivity analysis can help detect redundant constraints and identify alternative optima; therefore, it answers the question about what can be done with degenerate solutions.

Silicon Chip Corporation: Summary

Silicon Chip Corporation: Problem description

A Silicon Valley firm specializes in making four types of silicon chips for personal computers. Each chip must go through four stages of processing before completion. First the basic silicon wafers (100 chips each) are manufactured, second the wafers are laser etched with a micro circuit, next the circuit is laminated onto the chip, and finally the chip is tested and packaged for shipping. The production manager desires to maximize profits during the next month. During the next 30 days she has enough raw material to produce 4000 silicon wafers. Moreover, she has 600 hours of etching time, 900 hours of lamination time, and 700 hours of testing time. Taking into account depreciated capital investment, maintenance costs, and the cost of labor, each raw silicon wafer is worth $1, each hour of etching time costs $40, each hour of lamination time costs $60, and each hour of inspection time costs $10. The production manager has formulated her problem as a profit maximization

Silicon Chip Corporation: Problem conceptualization

- Four types of chips (types 1–4). Each batch is 100 chips.
- Four processing stages: raw wafers, etching, lamination, testing.
- Next 30 days availability:
  - Raw wafers: 4000 units
  - Etching: 600 hours
  - Lamination: 900 hours
  - Testing: 700 hours
- Costs per resource (depreciated etc.): wafer \$1 each, etching \$40/hr, lamination \$60/hr, testing \$10/hr.
- Objective: maximize profit.

Decision variables and initial tableau

Let x_i = number of 100-chip batches of type i (for $i = 1, \ldots, 4$). The initial tableau (coefficients per 100-chip batch):

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	b
raw wafers	100	100	100	100	1	0	0	0	4000
etching	10	10	20	20	0	1	0	0	600
lamination	20	20	30	20	0	0	1	0	900
testing	20	10	30	30	0	0	0	1	700
profit (per 100)	-2000	-3000	-5000	-4000	0	0	0	0	0

(Here $x_5, \ldots, x_8$ are slack variables.)

Optimal solution (given)

The (given) optimal tableau after simplex pivots is:

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	b
	0.5	1	0	0	0.015	0	0	-0.05	25
	-5	0	0	0	-0.05	1	0	-0.5	50
	0	0	1	0	-0.02	0	0.1	0	10
	0.5	0	0	1	0.015	0	-0.1	0.05	5
z	1500	0	0	0	5	0	100	50	145000

So the optimal production schedule in batches (100-chip units) is

$$(x_1, x_2, x_3, x_4) = (0, 25, 10, 5),$$

with optimal profit $145,000.

Interpreting the objective-row entries

- For a maximization problem in this tableau-sign convention, the objective row entries under nonbasic variables are $\Delta = c - A^T y$ (up to sign convention). Negative entries indicate *reduced costs*.
- Example: the entry under x_1 is 1500 $\Rightarrow$ reduced cost $= 1500$, so producing one 100-batch of type 1 would reduce z by \$1500.
- Break-even sale price adds this reduced cost to the current sale price to find the price at which the activity becomes profitable.

Changing an objective coefficient: general pivot form: Summary

Simplex pivots as left multiplication

The initial augmented matrix is

$$\begin{bmatrix} A & I & b \\ c^T & 0 & 0 \end{bmatrix}.$$

Pivoting to an optimal tableau corresponds to left-multiplication by a matrix

$$G = \begin{bmatrix} R & 0 \\ -y^T & 1 \end{bmatrix},$$

where R is the nonsingular record (basis) matrix and y is the dual vector. The resulting tableau is

$$\begin{bmatrix} RA & R & Rb \\ c^T - y^T A & -y^T & -b^T y \end{bmatrix}.$$

In the final tableau $c - A^T y$ appears in the objective row.

Effect of changing objective coefficients

If c changes to $c + \Delta c$, performing the same left multiplication gives

$$\begin{bmatrix} RA & R & Rb \\ (c+\Delta c)^T - y^T A & -y^T & -b^T y \end{bmatrix} = \begin{bmatrix} RA & R & Rb \\ (c - A^T y)^T + \Delta c^T & -y^T & -b^T y \end{bmatrix}$$

So the new objective-row entries are obtained by adding Δc to the old objective row. The current basis remains optimal if the modified objective-row entries keep the sign conditions (non-positive here).

Adding θ to c_1: $2000 + \theta$ scenario: Summary

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Modify c_1 to $2000 + \theta$

Replace the first objective coefficient by $2000 + \theta$. This corresponds to $\Delta c = \theta e_1$. Applying the previous observation, the optimal tableau's objective row becomes

$$(\text{old objective row}) + \theta e_1.$$

Using the numeric optimal tableau earlier, the modified objective row (bottom row) has $-1500 + \theta$ under x_1. Thus to preserve optimality we require

$$-1500 + \theta \leq 0 \quad \Rightarrow \quad \theta \leq 1500.$$

Hence break-even occurs at $\theta = 1500$, matching the reduced-cost interpretation.

Representative tableau with $2000 + \theta$

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	b
rows	0.5	1	0	0	0.015	0	0	-0.05	25
	-5	0	0	0	-0.05	1	0	-0.5	50
	0	0	1	0	-0.02	0	0.1	0	10
	0.5	0	0	1	0.015	0	-0.1	0.05	5
z	$-1500 + \theta$	0	0	0	-5	0	-100	-50	145000

Condition for current basis to remain optimal: $-1500 + \theta \leq 0$ (and other objective-row entries non-positive, which they already are).

Break-even Prices and Reduced Costs: Summary

Break-even price — intuition

- A decision variable not in the basis (nonbasic) has an **objective row coefficient** in the final tableau.
- The **reduced cost** of a nonbasic variable equals the negative of that objective-row coefficient (when objective expressed as $z - (\cdots) = 0$).
- Reduced cost = how much the objective coefficient must increase before the variable enters the basis.

Example: Type 1 chip break-even price

Current per-batch profit shown in original LP: \$2000 per 100-unit batch for type 1. Compute production cost per 100-batch:

$$\begin{aligned}
\text{chip cost} &= 100 \times 1 = 100 \\
\text{etching cost} &= 10 \times 40 = 400 \\
\text{lamination cost} &= 20 \times 60 = 1200 \\
\text{inspection cost} &= 20 \times 10 = 200 \\
\text{total cost per batch} &= 1900
\end{aligned}$$

So current sale price per batch $=$ profit $+$ cost $= 2000 + 1900 = \$3900$, i.e. \$39 per chip.

Type 1 reduced cost and break-even

The reduced cost of a decision variable is the needed increase in its objective row coefficient in order for it to be included in the optimal solution. For non-basic variables the break-even sale price can be read off from the reduced costs in the optimal tableau.
In the optimal tableau the objective row entry under x_1 equals -1500 (objective row entry is -1500 in the printed tableau). That means producing one batch of type 1 reduces optimal z by \$1500. To make x_1 profitable, the sale price must increase by \$1500 per 100-batch, i.e. \$15 per chip. So break-even sale price per chip $= 39 + 15 = \$54$.

More intuitive explanation

Now consider a more intuitive and simpler explanation of break-even sale prices. One way to determine these prices, is to determine by how much our profit is reduced if we produce one batch of these chips. Recall that the objective row coefficients in the optimal tableau correspond to the following expression for the objective variable z: $z = 145000 - 1500x1 - 5x5 - 100x7 - 50x8$. Hence, if we make one batch of type 1 chip, we reduce our optimal value by \$1500. Thus, to recoup this loss we must charge \$1500 more for these chips yielding a break-even sale price of \$39 + \$15 = \$54 per chip.

Range Analysis for Objective Coefficients: Summary

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Range analysis: goal

Range analysis is a tool for understanding the effects of both objective coefficient variations as well as resource availability variations.

- For a basic variable, find interval of objective coefficient values for which the current basis stays optimal.
- For a nonbasic variable, find how much objective coefficient can change before it becomes profitable to enter the basis.

Range analysis: method (summary)

- Changing c (objective coefficients) to $c + \Delta c$ results in adding Δc to the objective row of the current tableau after applying the same pivot record matrix.
- If the modified objective row (for nonbasic columns) remains non-positive (for a maximization LP in this tableau convention), the current basis remains optimal.
- Solve inequalities on the modified objective row entries to get allowable ranges.

Range for c_3 (type 3 chips)

Let $c_3 = 5000 + \theta$. The objective row after adjusting gives (reduced) modifications that must satisfy:

$$\begin{aligned} -5 + 0.02\theta \leq 0 \quad &\Rightarrow \quad \theta \leq 250, \\ -100 - 0.1\theta \leq 0 \quad &\Rightarrow \quad \theta \geq -1000. \end{aligned}$$

Thus

$$-1000 \leq \theta \leq 250 \quad \Rightarrow \quad 4000 \leq c_3 \leq 5250.$$

So the basis $\{x_2, x_3, x_4, x_6\}$ remains optimal while c_3 stays in this interval.

Perturb $c_3 = 5000 + \theta$ (type 3 in basis)

Start with $c_3 = 5000 + \theta$. After adding θe_3 to the objective row of the current tableau, we obtain a tableau with an extra θ under column x_3 in the objective row:

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	b
(rows)	0.5	1	0	0	0.015	0	0	-0.05	25
	-5	0	0	0	-0.05	1	0	-0.5	50
	0	0	1	0	-0.02	0	0.1	0	10
	0.5	0	0	1	0.015	0	-0.1	0.05	5
z	-1500	0	θ	0	-5	0	-100	-50	145000

This is not a proper tableau (because of nonzero objective entry under a basic column); we must pivot to eliminate θ (i.e., adjust by multiples of the basic row corresponding to x_3).

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Eliminate θ by adding $-\theta$ times row 3 to the objective row

Multiply row 3 (the basic row for x_3) by $-\theta$ and add to the objective row:

$$z_{\text{new}} = z_{\text{old}} - \theta \cdot \text{row}_3.$$

This changes several objective-row entries; with the numeric row entries we get (representative):

z	-1500	0	0	0	$-5 + 0.02\theta$	0	$-100 - 0.1\theta$	-50	$145000 - 10\theta$

To preserve optimality all entries that correspond to nonbasic variables must remain non-positive:

$$-5 + 0.02\theta \leq 0, \quad -100 - 0.1\theta \leq 0, \quad \ldots$$

Solving yields:

$$-1000 \leq \theta \leq 250 \quad \Rightarrow \quad 4000 \leq c_3 \leq 5250.$$

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Range for c_4 (type 4 chips) and c_2

A similar perturbation for $c_4 = 4000 + \theta$ yields inequalities (from objective row entries) which give:

$$-333.\overline{3} \leq \theta \leq 1000,$$

so

$$3666.\overline{6} \leq c_4 \leq 5000.$$

For c_2 (originally 3000) the derived range is

$$1666.\overline{6} \leq c_2 \leq 3000.$$

(These follow from ensuring all objective-row entries remain non-positive.)

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Perturb $c_4 = 4000 + \theta$

A similar calculation for c_4 yields (after appropriate row operations and algebra) the inequalities:

$$-333.\overline{3} \leq \theta \leq 1000,$$

giving

$$3666.\overline{6} \leq c_4 \leq 5000.$$

(These bounds come from ensuring all objective-row entries remain non-positive after the adjustments.)

Range for c_2 (summary)

For c_2 the lecture's algebra gives:

$$-333.\overline{3} \leq \theta \leq 1000 \quad \Rightarrow \quad 1666.\overline{6} \leq c_2 \leq 3000$$

(Again, derived from the sign constraints on the objective row after adding Δc and pivoting as needed.)

Resource Variations, Marginal Values, and Range Analysis: Summary

Resource variation questions

Suppose we change a right-hand-side (RHS) resource (e.g., purchase more wafers). Typical questions:

1. How many additional units should we buy?
2. What is the most we should pay per unit?
3. What is the new optimal schedule after purchasing?

The dual variables (shadow prices) provide the marginal value of resources for small changes (within allowable intervals).

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RHS perturbation and marginal values (sketch)

- Let b be the RHS vector and suppose basis B is optimal with record matrix R.
- The basic solution is $x_B = Rb$. If b changes by Δb, the basic solution becomes $x_B = R(b + \Delta b)$.
- If x_B remains feasible (nonnegative), the basis remains feasible; combined with dual feasibility, the basis remains optimal.
- The dual solution y remains the same for small enough changes; the change in optimal objective is $y^\top \Delta b$.

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Right Hand Side Perturbations: Summary

Allowable RHS changes (concept)

- For each basic variable, determine how far the RHS can move until a basic variable becomes zero (i.e., feasibility violated).
- These bounds yield intervals for each RHS entry within which the current basis remains optimal.
- Shadow prices hold within these intervals.

Example: marginal value interpretation

If the dual variable (shadow price) associated with raw wafers is π_1, then:

$$\text{small increase of 1 wafer} \Rightarrow \text{increase in objective} \approx \pi_1.$$

For finite changes, use range analysis to confirm how large a change preserves the basis (so π_1 stays valid).

Pricing Out: Summary

Pricing out (brief)

- When a resource is extremely expensive (or cheap) it may be optimal to remove it (or saturate it).
- Analyze by varying the RHS or adding cost to resource availability to see when shadow prices change sign or basis changes.

The Fundamental Theorem on Sensitivity Analysis: Summary

Fundamental theorem (informal statement)

Given a linear program with current optimal basis B, there exist non-empty intervals for each objective coefficient and each RHS entry such that any changes to those coefficients within their respective intervals preserve the optimal basis. Within those intervals:

- The dual solution stays constant.
- The objective value changes linearly with the perturbations (via the dual for RHS, via basic solution for objective-row changes).

Practical takeaways

- Reduced costs read directly from the final tableau give break-even adjustments for nonbasic variables.
- Adding Δc to c simply adds Δc to the objective row of the final tableau (then pivot if necessary).
- To analyze RHS changes use $x_B = Rb$ and dual vector y; objective change $\Delta z = y^\top \Delta b$ while the basis remains valid.

Practical takeaways

- Binding constraint $\Rightarrow$ LHS = RHS.
- Shadow price = 0 $\Rightarrow$ constraint not binding.
- If an objective coefficient changes outside the allowable interval, the optimal solution may change.
- Shadow prices indicate how the objective changes with small changes in the RHS.
- Use reduced costs to read break-even prices for nonbasic variables.
- Use range analysis to find intervals where the basis remains optimal (both for c and b).
- Use shadow prices (dual variables) for marginal valuations of resources — valid for small changes within allowable intervals.
- When changes fall outside allowable intervals, recompute basis (re-run LP) or update tableau with pivots.

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Some tools that help interpreting the calculations

Find *here* a code with the solution using ORtools.

The Sensitivity analysis tool in Excel is nicely explained *here*.

Notice that in a non-linear scenario, what we have identified here as *reduced costs* are in fact *reduced gradient* (or actual gradient of the optimal solution) while *shadow prices* (margin for profit to be obtained by changing 1 unit in each constraint RHS term) correspond to the *Lagrange multipliers*.

References I

[1] Michael Carter, Camille C Price, and Ghaith Rabadi. *Operations Research. A Practical Introduction. Second Edition*. CRC Press, 2019. ISBN: 978-1-4987-8010-0.

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