Unit 7. Stochastic processes

Jordi Villà i Freixa

Universitat de Vic - Universitat Central de Catalunya
Study Abroad. Operations Research

jordi.villa@uvic.cat
https://mon.uvic.cat/cbbl

last updated: December 16, 2025

December 10-17th, 2025

Preliminary

This course is strongly based on the monography on Operations Research by Carter, Price and Rabadi [1], and in material obtained from different sources (quoted when needed through the slides).

Markov chains: Summary

State Transitions

A Markov process is a discrete-time stochastic process in which the next state depends only on the current state:

$$P(X_{t+1} = s_{t+1} \mid X_t = s_t) = P(X_{t+1} = s_{t+1} \mid X_t = s_t, \ldots, X_0 = s_0).$$

- Finite number of states: $1, \ldots, N$
- System occupies one state at each time
- Transitions occur randomly according to probabilities

State Probabilities

Let $p^{(0)}$ be the initial state probability vector. State probabilities evolve as:

$$p^{(1)} = p^{(0)}P, \quad p^{(2)} = p^{(0)}P^2, \ldots$$

To compute n-step transition probabilities:

$$P^{(n)} = P^n.$$

Examples include:

- computing probability of a sequence of weather conditions
- identifying that probabilities tend to stabilize in many cases

First Passage Probabilities

First passage probability $f_{ij}^{(n)}$ is the probability that the process enters j for the first time exactly at step n.
Properties:

$$p_{ij}^{(n)} = f_{ij}^{(n)} + \sum_k f_{ij}^{(k)} p_{jj}^{(n-k)}.$$

Used to compute:

- probability of reaching a state for the first time
- probability of return to a state after some steps

Properties of States

- **Reachable:** j reachable from i if $p_{ij}^{(n)} > 0$ for some n
- **Irreducible:** every state reachable from every other state
- **Closed set:** no transitions from the set to outside
- **Absorbing state:** once entered, never left $(p_{ii} = 1)$
- **Transient state:** possible to leave forever
- **Periodic state:** returns occur only at multiples of some $t > 1$
- **Ergodic chain:** irreducible and aperiodic

Steady-State Analysis

For an ergodic chain:

$$P^n \to \begin{bmatrix} \pi \\ \pi \\ \vdots \\ \pi \end{bmatrix}, \quad \pi P = \pi, \quad \sum \pi_i = 1.$$

Interpretation:

- $\pi_i =$ long-run probability of being in state i
- independent of initial state

FACULTAT DE CIÈNCIES I TECNOLOGIA UVIC | UVIC-UCC

Expected First Passage Times

First passage time T_{ij}: steps needed to reach j from i.
For $i \neq j$:

$$m_{ij} = 1 + \sum_k p_{ik} m_{kj}.$$

For recurrence:

$$m_{ii} = \frac{1}{\pi_i}.$$

Applications include computing expected times between weather changes.

Absorbing Chains

A Markov chain with at least one absorbing state.
Reorder states so transient states come first:

$$P = \begin{bmatrix} Q & R \\ 0 & I \end{bmatrix}.$$

- Q: transitions among transient states
- R: transitions from transient to absorbing

Fundamental Matrix and Absorption Probabilities

Fundamental matrix:

$$F = (I - Q)^{-1}.$$

Meaning:

- element f_{ij} = expected number of visits to transient state j starting from i

Absorption probabilities:

$$A = F\,R.$$

Water Reservoir Operations

- States represent storage volumes
- Transitions derived from historical inflow data and release decisions
- Objectives: flood control, power generation, low-season flow
- Markov analysis improves long-term expected reward and performance

Dynamic Memory Allocation

- States represent sequences of allocated/free memory block lengths
- Transition matrix derived from allocation/release rules
- Steady-state probabilities used to evaluate memory usage
- Applicable to “first-fit”, “best-fit”, “worst-fit”, buddy systems

Manufacturing Production Capability

- States track failures in a multi-unit system
- Elements may be catastrophic, dependent, or independent
- Steady-state analysis gives:
  - expected production rate
  - probability of shutdown
  - optimal repair policies

Exercise 1

Consider the following simple betting game (adapted from Wikibooks):

- The player wagers \$1 on each fair coin toss.
- The game stops when the player's capital reaches \$0 or \$5.

If the player starts with \$3, what is the probability that the game lasts at least 5 tosses? At least 25 tosses?

At any time the player's capital is \$0, \$1, ..., \$5, which we denote by states $s_0, \ldots, s_5$. From an interior state s_i (with $1 \leq i \leq 4$) the process moves to s_{i-1} or s_{i+1} with probability $1/2$ each. States s_0 and s_5 are absorbing (once reached the process stays there).

FACULTAT DE CIÈNCIES I TECNOLOGIA UVIC | UVIC-UCC

Let $p_i(n)$ denote the probability that the player is in state s_i after n flips. For example,

$$p_0(n+1) = p_0(n) + \frac{1}{2} p_1(n).$$

In matrix form this becomes

$$\begin{pmatrix} 1 & .5 & 0 & 0 & 0 & 0 \\ 0 & 0 & .5 & 0 & 0 & 0 \\ 0 & .5 & 0 & .5 & 0 & 0 \\ 0 & 0 & .5 & 0 & .5 & 0 \\ 0 & 0 & 0 & .5 & 0 & 0 \\ 0 & 0 & 0 & 0 & .5 & 1 \end{pmatrix} \begin{pmatrix} p_0(n) \\ p_1(n) \\ p_2(n) \\ p_3(n) \\ p_4(n) \\ p_5(n) \end{pmatrix} = \begin{pmatrix} p_0(n+1) \\ p_1(n+1) \\ p_2(n+1) \\ p_3(n+1) \\ p_4(n+1) \\ p_5(n+1) \end{pmatrix}$$

Assuming the player starts with \$3 (state s_3), the state probability vector after n tosses is

$$p^{(n)} = p^{(0)} P^n, \qquad p^{(0)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

The table below lists $p^{(0)}, p^{(1)}, \ldots, p^{(24)}$.

$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$\cdots$	$n=24$
$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \\ .5 \\ 0 \\ .5 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ .25 \\ 0 \\ .5 \\ 0 \\ .25 \end{pmatrix}$	$\begin{pmatrix} .125 \\ 0 \\ .375 \\ 0 \\ .25 \\ .25 \end{pmatrix}$	$\begin{pmatrix} .125 \\ .1875 \\ 0 \\ .3125 \\ 0 \\ .375 \end{pmatrix}$	$\cdots$	$\begin{pmatrix} .39600 \\ .00276 \\ 0 \\ .00447 \\ 0 \\ .59676 \end{pmatrix}$

- The process is quickly absorbed into either s_0 or s_5; for example, after four flips there is a probability of 0.50 that the game has already ended. These boundary states are absorbing.
- Memoryless (Markov) property:

$$P\big(X_{t+1} = s_{t+1} \mid X_t = s_t, X_{t-1} = s_{t-1}, \ldots, X_0 = s_0\big) = P\big(X_{t+1} = s_{t+1} \mid$$

- Stationarity: transition probabilities do not depend on time,

$$P\big(X_{t+1} = s_j \mid X_t = s_i\big) = p_{ij} \quad \text{for all } t.$$

- Each row of the transition matrix P is a probability vector:

$$\sum_j p_{ij} = 1 \quad \text{for every state } i,$$

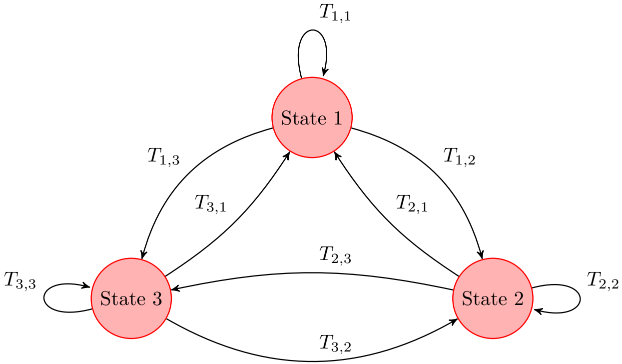
so P is a (row-)stochastic matrix.

Exercise 2

A study classified occupations in the United Kingdom into three groups—upper (executives and professionals), middle (supervisors and skilled manual workers), and lower (unskilled) [2]. About two thousand men were asked to report their own occupation and the occupation of their father when the respondent was fourteen, to measure intergenerational occupational mobility.

This equation and transition diagram summarizes the results.

$$\begin{pmatrix} T_{1,1} & T_{1,2} & T_{1,3} \\ T_{2,1} & T_{2,2} & T_{2,3} \\ T_{3,1} & T_{3,2} & T_{3,3} \end{pmatrix}^t \rightarrow \begin{pmatrix} .60 & .29 & .16 \\ .26 & .37 & .27 \\ .14 & .34 & .57 \end{pmatrix} \begin{pmatrix} p_U(n) \\ p_M(n) \\ p_L(n) \end{pmatrix} = \begin{pmatrix} p_U(n+1) \\ p_M(n+1) \\ p_L(n+1) \end{pmatrix}$$



For example, a child of a lower–class worker has a 0.27 probability of becoming middle class. The Markov assumption—that the next state depends only on the current state—seems reasonable here: a parent's occupation directly influences the child's outcome, whereas any grandparent effect is largely indirect. Using the initial distribution of the respondents' fathers given below, the table shows the occupational distributions for the next five generations.

$n = 0$	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$
$\begin{pmatrix} .12 \\ .32 \\ .56 \end{pmatrix}$	$\begin{pmatrix} .23 \\ .34 \\ .42 \end{pmatrix}$	$\begin{pmatrix} .29 \\ .34 \\ .37 \end{pmatrix}$	$\begin{pmatrix} .31 \\ .34 \\ .35 \end{pmatrix}$	$\begin{pmatrix} .32 \\ .33 \\ .34 \end{pmatrix}$	$\begin{pmatrix} .33 \\ .33 \\ .34 \end{pmatrix}$

Exercise 3

The World Series is played between the American League (AL) and National League (NL) champions. The first team to win four games wins the series, so the contest can be represented by one of 24 states: 0–0, 1–0, ..., 4–3, where a–b denotes a wins for the AL team and b for the NL team (states with a 4 are absorbing). Assuming each game is independently won by the AL team with probability p, the sequence of states is a Markov chain with the following transition matrix. Adapted from Wikibooks.

$$\begin{pmatrix} 0 & 0 & 0 & 0 & \dots \\ p & 0 & 0 & 0 & \dots \\ 1-p & 0 & 0 & 0 & \dots \\ 0 & p & 0 & 0 & \dots \\ 0 & 1-p & p & 0 & \dots \\ 0 & 0 & 1-p & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix} \begin{pmatrix} p_{0\text{-}0}(n) \\ p_{1\text{-}0}(n) \\ p_{0\text{-}1}(n) \\ p_{2\text{-}0}(n) \\ p_{1\text{-}1}(n) \\ p_{0\text{-}2}(n) \\ \vdots \end{pmatrix} = \begin{pmatrix} p_{0\text{-}0}(n+1) \\ p_{1\text{-}0}(n+1) \\ p_{0\text{-}1}(n+1) \\ p_{2\text{-}0}(n+1) \\ p_{1\text{-}1}(n+1) \\ p_{0\text{-}2}(n+1) \\ \vdots \end{pmatrix}$$

An especially interesting special case is $p = 0.50$; this table lists the resulting components of the $n = 0$ through $n = 7$ vectors. Check what happens with other values of p.

	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$
$0-0$	1	0	0	0	0	0	0	0
$1-0$	0	0.5	0	0	0	0	0	0
$0-1$	0	0.5	0	0	0	0	0	0
$2-0$	0	0	0.25	0	0	0	0	0
$1-1$	0	0	0.5	0	0	0	0	0
$0-2$	0	0	0.25	0	0	0	0	0
$3-0$	0	0	0	0.125	0	0	0	0
$2-1$	0	0	0	0.375	0	0	0	0
$1-2$	0	0	0	0.375	0	0	0	0
$0-3$	0	0	0	0.125	0	0	0	0
$4-0$	0	0	0	0	0.0625	0.0625	0.0625	0.0625
$3-1$	0	0	0	0	0.25	0	0	0
$2-2$	0	0	0	0	0.375	0	0	0
$1-3$	0	0	0	0	0.25	0	0	0
$0-4$	0	0	0	0	0.0625	0.0625	0.0625	0.0625
$4-1$	0	0	0	0	0	0.125	0.125	0.125
$3-2$	0	0	0	0	0	0.3125	0	0
$2-3$	0	0	0	0	0	0.3125	0	0
$1-4$	0	0	0	0	0	0.125	0.125	0.125
$4-2$	0	0	0	0	0	0	0.15625	0.15625
$3-3$	0	0	0	0	0	0	0.3125	0
$2-4$	0	0	0	0	0	0	0.15625	0.15625
$4-3$	0	0	0	0	0	0	0	0.15625
$3-4$	0	0	0	0	0	0	0	0.15625

Evenly-matched teams are likely to have a long series– there is a probability of 0.625 that the series goes at least six games.

In summary, a system can be modeled as a Markov process if it has the following four properties:

- Property 1: A finite number of states can be used to describe the dynamic behavior of the system.
- Property 2: Initial probabilities are specified for the system.
- Property 3: Markov property—We assume that a transition to a new state depends only on the current state and not on past conditions.
- Property 4: Stationarity property—The probability of a transition between any two states does not vary in time.

If Markov analysis is appropriate for the system being studiedone can try answering questions like:

- How many transitions (steps) will it likely take for the system to move from some specified state to another specified state?
- What is the probability that it will take some given number of steps to go from one specified state to another?
- In the long run, which state is occupied by the system most frequently?
- Over a long period of time, what fraction of the time does the system occupy each of the possible states?
- Will the system continue indefinitely to move among all N states, or will it eventually settle into a certain few states?

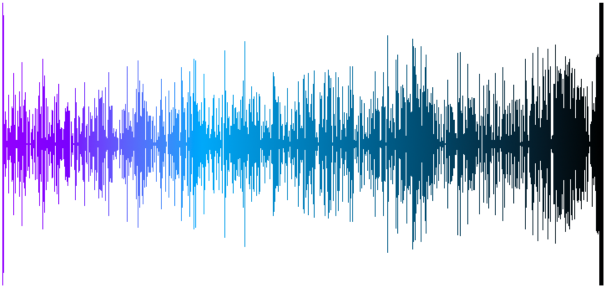
Hidden Markov Models: Summary

Hidden Markov Models

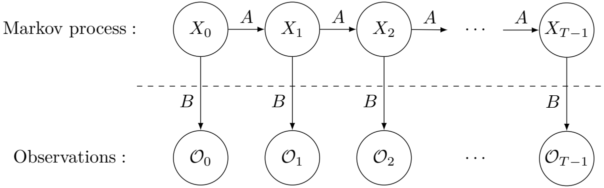
HMMs are probably the most popular directed graphical model out there. They are used in many sequential and temporal domains:

- speech recognition,
- handwriting recognition,
- visual target tracking and localization,
- machine translation,
- robot localization,
- gene prediction,
- time-series analysis,
- natural language processing and part-of-speech recognition,
- stochastic control,
- protein folding, ...

In HMMs, the random variables are divided into hidden states (phonemes, letters, target location) and observations (audio signal, pen strokes, target image). The goal is to predict the states from observations.
What is this?



We can start wth the graphical representation of the HMM



HMM

An HMM is defined by:

- A set of N hidden states $S = \{s_1, s_2, \ldots, s_N\}$.
- A set of M observation symbols $O = \{o_1, o_2, \ldots, o_M\}$.
- State transition probability distribution $A = \{a_{ij}\}$, where $a_{ij} = P(q_{t+1} = s_j | q_t = s_i)$.
- Observation symbol probability distribution $B = \{b_j(k)\}$, where $b_j(k) = P(v_k | q_t = s_j)$.
- Initial state distribution $\pi = \{\pi_i\}$, where $\pi_i = P(q_1 = s_i)$.

The complete parameter set of the model is $\lambda = (A, B, \pi)$.

References I

[1] Michael Carter, Camille C Price, and Ghaith Rabadi. *Operations Research. A Practical Introduction. Second Edition*. CRC Press, 2019. ISBN: 978-1-4987-8010-0.

[2] A. H. Halsey, ed. *British Social Trends since 1900*. London: Palgrave Macmillan UK, 1988. ISBN: 978-0-333-34522-1 978-1-349-19466-7. DOI: 10.1007/978-1-349-19466-7. URL: http://link.springer.com/10.1007/978-1-349-19466-7.